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MECHANICS AND THERMODYNAMICS OF A MIXTURE OF A GRANULAR MATERIAL WITH A FLUID

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Introduction

In a previous work [1] balance equations for a mixture of an arbitrary finite number of granular materials has been given; however, no constitutive theory has been developed. In this work I consider the special case of two materials, one a granular material as defined by Goodman and Cowin [2], the other a viscous fluid. A constitutive theory is postulated, and restrictions due to the entropy inequality are explored, following closely the analysis of Müller [3].

I use without further comment the equations and notations of [1]. Furthermore, sections and equations herein are numbered as if this work were a continuation of [1].

5. Further Analysis of the Entropy Inequality

Recall the equations for balance of energy and entropy for the mixture:

$$\rho\dot{\epsilon} = \mathrm{tr}(\underset{\sim}{T}^{T}\,\mathrm{grad}\,\dot{\underset{\sim}{x}}) + \underset{\sim}{h}\cdot\mathrm{grad}\,\dot{\nu} + \rho k\dot{\nu}^{2} + \rho g\dot{\nu} + \mathrm{div}\,\underset{\sim}{q} + \rho s\,, \tag{5.1}$$

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$$\rho\dot{\eta} \geq \operatorname{div} \underset{\sim}{\phi} + \rho\sigma\vartheta , \tag{5.2}$$

where

$$\rho s = \Sigma \underset{a}{\rho} \underset{a}{s} + \Sigma \underset{a}{\rho} [\underset{\sim a}{b} \cdot \underset{\sim a}{u} + \ell (\underset{a}{\grave{\nu}} - \dot{\nu})] , \tag{5.3}$$

$$\rho \vartheta\sigma = \Sigma \underset{a}{\rho} \underset{a}{\vartheta} \underset{a}{s} . \tag{5.4}$$

Assume a common coldness for each component*

$$\underset{a}{\vartheta} = \vartheta . \tag{5.5}$$

Then

$$\rho\sigma = \Sigma \underset{a}{\rho} \underset{a}{s}$$

and

$$\rho s = \rho\sigma + \Sigma \underset{a}{\rho} [\underset{\sim a}{b} \cdot \underset{\sim a}{u} + \underset{a}{\ell} (\underset{a}{\grave{\nu}} - \dot{\nu})] , \tag{5.7}$$

so that (5.1) becomes

$$\begin{aligned} \rho\dot{\epsilon} &= tr(\underset{\sim}{T}^{T} \operatorname{grad} \dot{\underset{\sim}{x}}) + \underset{\sim}{h} \cdot \operatorname{grad} \dot{\nu} + \rho \hat{k} \dot{\nu}^{2} + \rho g \dot{\nu} \\ &+ \operatorname{div} \underset{\sim}{q} + \rho\sigma + \Sigma \underset{a}{\rho} [\underset{\sim a}{b} \cdot \underset{\sim a}{u} + \underset{a}{\ell} (\underset{a}{\grave{\nu}} - \dot{\nu})] . \end{aligned} \tag{5.8}$$

---

*There is for the special case considered here a physical argument indicating that this assumption is too strong.

Eliminating $\rho\sigma$ between (5.8) and (5.2) gives

$$\begin{aligned}\rho\dot{\eta} \geq \operatorname{div}\boldsymbol{\phi} + \vartheta[\rho\dot{\epsilon} - \operatorname{tr}(\mathbf{T}^T\operatorname{grad}\dot{\mathbf{x}}) - \mathbf{h}\cdot\operatorname{grad}\dot{\nu} - \rho k\dot{\nu}^2 & \\ - \rho g\dot{\nu} - \operatorname{div}\mathbf{g} - \Sigma\rho_\alpha[\mathbf{b}_\alpha\cdot\mathbf{u}_\alpha + \ell_\alpha(\grave{\nu}_\alpha - \dot{\nu})]] \,. &\end{aligned} \tag{5.9}$$

By the linear momentum balance for a constituent

$$\rho_\alpha\mathbf{b}_\alpha\cdot\mathbf{u}_\alpha = \rho_\alpha\breve{\mathbf{x}}_\alpha\cdot\mathbf{u}_\alpha + \rho_\alpha\grave{\mathbf{x}}_\alpha\overset{+}{c}_\alpha\cdot\mathbf{u}_\alpha - \mathbf{u}_\alpha\cdot\operatorname{div}\mathbf{T}_\alpha - \rho\overset{+}{\mathbf{m}}_\alpha\cdot\mathbf{u}_\alpha, \tag{5.10}$$

also by the balance of equilibrated force (2.21),

$$\begin{aligned}\rho_\alpha\ell_\alpha(\grave{\nu}_\alpha - \dot{\nu}) = \rho_\alpha k_\alpha\breve{\nu}_\alpha(\grave{\nu}_\alpha - \dot{\nu}) + \rho_\alpha\grave{k}_\alpha\grave{\nu}_\alpha(\grave{\nu}_\alpha - \dot{\nu}) + \rho_\alpha k_\alpha\grave{\nu}_\alpha\overset{+}{c}_\alpha(\grave{\nu}_\alpha - \dot{\nu}) - (\grave{\nu}_\alpha - \dot{\nu})\operatorname{div}\mathbf{h}_\alpha & \\ - \rho_\alpha g_\alpha(\grave{\nu}_\alpha - \dot{\nu}) - \rho\overset{+}{v}_\alpha(\grave{\nu}_\alpha - \dot{\nu}) \,. &\end{aligned} \tag{5.11}$$

Define

$$\hat{\boldsymbol{\phi}} = \boldsymbol{\phi} - \vartheta\mathbf{g} + \vartheta\Sigma\mathbf{T}_\alpha^T\mathbf{u}_\alpha + \vartheta\Sigma\mathbf{h}_\alpha(\grave{\nu}_\alpha - \dot{\nu}), \tag{5.12}$$

so that

$$\begin{aligned}\operatorname{div}\boldsymbol{\phi} = \operatorname{div}\hat{\boldsymbol{\phi}} + \mathbf{g}\cdot\operatorname{grad}\vartheta + \vartheta\operatorname{div}\mathbf{g} & \\ - \Sigma\mathbf{T}_\alpha^T\mathbf{u}_\alpha\cdot\operatorname{grad}\vartheta - \vartheta\Sigma\mathbf{u}_\alpha\cdot\operatorname{div}\mathbf{T}_\alpha - \vartheta\Sigma\operatorname{tr}(\mathbf{T}_\alpha^T\operatorname{grad}\mathbf{u}_\alpha) & \\ - \Sigma\mathbf{h}_\alpha(\grave{\nu}_\alpha - \dot{\nu})\cdot\operatorname{grad}\vartheta - \vartheta\Sigma(\grave{\nu}_\alpha - \dot{\nu})\operatorname{div}\mathbf{h}_\alpha - \vartheta\Sigma\mathbf{h}_\alpha\cdot\operatorname{grad}(\grave{\nu}_\alpha - \dot{\nu}) \,. &\end{aligned} \tag{5.13}$$

Also, the Helmholtz free energy of the mixture is

$$\psi = \epsilon - \frac{\eta}{\vartheta} . \tag{5.14}$$

Eliminating $\underset{\alpha}{\mathbf{b}}$ and $\underset{\alpha}{\ell}$ from (5.9), and using (5.13) and (5.14) gives, by use of $(2.25)_9$,

$$\begin{aligned}
\rho\eta\frac{\dot{\vartheta}}{\vartheta} - \rho\vartheta\dot{\psi} \geq \operatorname{div}\hat{\boldsymbol{\phi}} &- \vartheta\Big[\Sigma \underset{\alpha}{\rho}\underset{\alpha}{\bar{\mathbf{x}}}\cdot\underset{\alpha}{\mathbf{u}} + \Sigma\underset{\alpha}{\rho}(\underset{\alpha}{k}\underset{\alpha}{\bar{\nu}} + \underset{\alpha}{\grave{k}}\underset{\alpha}{\grave{\nu}})(\underset{\alpha}{\grave{\nu}} - \dot{\nu})\Big] \\
&+ \Big[\mathbf{q} - \Sigma\underset{\alpha}{\mathbf{T}}^T\underset{\alpha}{\mathbf{u}} - \Sigma\underset{\alpha}{\mathbf{h}}(\underset{\alpha}{\grave{\nu}} - \dot{\nu})\Big]\cdot\operatorname{grad}\vartheta \\
&+ \vartheta\rho\Sigma(\underset{\alpha}{\overset{+}{\mathbf{m}}} - \underset{\alpha}{\overset{+}{c}}\underset{\alpha}{\grave{\mathbf{x}}})\cdot\underset{\alpha}{\mathbf{u}} + \vartheta\rho\Sigma(\underset{\alpha}{\overset{+}{v}} - \underset{\alpha}{\overset{+}{c}}\underset{\alpha}{k}\underset{\alpha}{\grave{\nu}})(\underset{\alpha}{\grave{\nu}} - \dot{\nu}) \\
&- \vartheta\Big[\operatorname{tr}(\mathbf{T}^T\operatorname{grad}\dot{\mathbf{x}}) + \Sigma\operatorname{tr}(\underset{\alpha}{\mathbf{T}}^T\operatorname{grad}\underset{\alpha}{\mathbf{u}}) + \mathbf{h}\cdot\operatorname{grad}\dot{\nu} + \Sigma\underset{\alpha}{\mathbf{h}}\cdot\operatorname{grad}(\underset{\alpha}{\grave{\nu}} - \dot{\nu})\Big] \\
&- \tfrac{1}{2}\rho k\dot{\nu}^2 - \Sigma\underset{\alpha}{\rho}\underset{\alpha}{g}\underset{\alpha}{\grave{\nu}}) .
\end{aligned} \tag{5.15}$$

Define

$$\mathbf{G} = \operatorname{grad}\dot{\mathbf{x}} , \tag{5.16}$$

the velocity gradient, and

$$\underset{\alpha}{\mathbf{G}} = \operatorname{grad}\underset{\alpha}{\grave{\mathbf{x}}} \tag{5.17}$$

the peculiar velocity gradient of the α-th constituent. The symmetric and skew parts of these are

$$\underset{\sim}{D} = \frac{1}{2}(\underset{\sim}{G} + \underset{\sim}{G}^T), \quad \underset{\sim}{W} = \frac{1}{2}(\underset{\sim}{G} - \underset{\sim}{G}^T), \tag{5.18}$$

the stretching and spin, and

$$\underset{\alpha}{\underset{\sim}{D}} = \frac{1}{2}(\underset{\alpha}{\underset{\sim}{G}} + \underset{\alpha}{\underset{\sim}{G}}^T), \quad \underset{\alpha}{\underset{\sim}{W}} = \frac{1}{2}(\underset{\alpha}{\underset{\sim}{G}} - \underset{\alpha}{\underset{\sim}{G}}^T), \tag{5.19}$$

the peculiar stretching and peculiar spin of the α-th constituent.

By (2.11)

$$\text{grad } \underset{\alpha}{\underset{\sim}{u}} = \underset{\alpha}{\underset{\sim}{G}} - \underset{\sim}{G}, \tag{5.20}$$

and by (1.6) and (1.10)

$$\rho \underset{\sim}{G} = \Sigma \underset{\alpha}{\rho} \underset{\alpha}{\underset{\sim}{G}} + \Sigma \underset{\alpha}{\underset{\sim}{u}} \otimes \text{grad } \underset{\alpha}{\rho}, \tag{5.21}$$

so that

$$\rho \underset{\sim}{D} = \Sigma \underset{\alpha}{\rho} \underset{\alpha}{\underset{\sim}{D}} + \frac{1}{2} \Sigma(\underset{\alpha}{\underset{\sim}{u}} \otimes \text{grad } \underset{\alpha}{\rho} + \text{grad } \underset{\alpha}{\rho} \otimes \underset{\alpha}{\underset{\sim}{u}}), \tag{5.22}$$

and

$$\rho \underset{\sim}{W} = \Sigma \underset{\alpha}{\rho} \underset{\alpha}{\underset{\sim}{W}} + \frac{1}{2} \Sigma(\underset{\alpha}{\underset{\sim}{u}} \otimes \text{grad } \underset{\alpha}{\rho} - \text{grad } \underset{\alpha}{\rho} \otimes \underset{\alpha}{\underset{\sim}{u}}). \tag{5.22}$$

By (2.19) and $(2.23)_3$ $\Sigma \underset{\alpha}{\underset{\sim}{T}}$ is symmetric, and by $(2.25)_1$ $\underset{\sim}{T}$ is symmetric. Then by (5.18) - (5.21),

$$
\begin{aligned}
&\operatorname{tr}(\mathbf{T}^T \operatorname{grad} \dot{\mathbf{x}}) + \Sigma \operatorname{tr}(\mathbf{T}_\alpha^T \operatorname{grad} \mathbf{u}_\alpha) \\
&\quad = \operatorname{tr} \mathbf{T}\mathbf{D} - \operatorname{tr}(\Sigma \mathbf{T}_\alpha)\mathbf{D} + \Sigma \operatorname{tr} \mathbf{T}_\alpha^T \mathbf{G}_\alpha \qquad (5.23) \\
&\quad = \operatorname{tr}(\mathbf{T} - \Sigma \mathbf{T}_\alpha)\Big(\Sigma \frac{\rho_\alpha}{\rho} \mathbf{D}_\alpha + \frac{1}{\rho} \mathbf{u}_\alpha \otimes \operatorname{grad} \rho_\alpha\Big) + \Sigma \operatorname{tr} \mathbf{T}_\alpha^T \mathbf{G}_\alpha
\end{aligned}
$$

The gradient of $(1.11)_2$ is

$$
(\operatorname{grad} \rho k)\dot{\nu} + \rho k \operatorname{grad} \dot{\nu} = \Sigma(\operatorname{grad} \rho_\alpha k_\alpha)\grave{\nu}_\alpha + \Sigma \rho_\alpha k_\alpha \operatorname{grad} \grave{\nu}_\alpha \,. \qquad (5.24)
$$

However, by (1.11),

$$
\operatorname{grad} \rho k = \Sigma \operatorname{grad} \rho_\alpha k_\alpha \,. \qquad (5.25)
$$

Substituting (5.25) into (5.24), I obtain the result

$$
\rho k \operatorname{grad} \dot{\nu} = \Sigma \rho_\alpha k_\alpha \operatorname{grad} \grave{\nu}_\alpha + \Sigma[\operatorname{grad}(\rho_\alpha k_\alpha)](\grave{\nu}_\alpha - \dot{\nu}) \,. \qquad (5.26)
$$

Then

$$
\begin{aligned}
&\mathbf{h} \cdot \operatorname{grad} \dot{\nu} + \Sigma \mathbf{h}_\alpha \cdot \operatorname{grad}(\grave{\nu}_\alpha - \dot{\nu}) \\
&\quad = (\mathbf{h} - \Sigma \mathbf{h}_\alpha) \cdot \operatorname{grad} \dot{\nu} + \Sigma \mathbf{h}_\alpha \cdot \operatorname{grad} \grave{\nu}_\alpha \qquad (5.27) \\
&\quad = (\mathbf{h} - \Sigma \mathbf{h}_\alpha) \cdot \frac{1}{\rho k}[\Sigma \rho_\alpha k_\alpha \operatorname{grad} \grave{\nu}_\alpha + \Sigma (\grave{\nu}_\alpha - \dot{\nu}) \operatorname{grad}(\rho_\alpha k_\alpha)] \\
&\qquad + \Sigma \mathbf{h}_\alpha \cdot \operatorname{grad} \grave{\nu}_\alpha \,.
\end{aligned}
$$

Substituting (5.23) and (5.2?) into (5.15), yields

$$
\begin{aligned}
\rho\eta\frac{\dot{\vartheta}}{\vartheta} - \rho\vartheta\dot{\psi} \geq{} & \operatorname{div}\hat{\boldsymbol{\Phi}} - \vartheta\Big[\Sigma\, \underset{\alpha}{\rho}\,\underset{\alpha}{\tilde{\mathbf{x}}}\cdot\underset{\alpha}{\mathbf{u}} + \Sigma\big(\underset{\alpha}{\rho}\,\underset{\alpha}{k}\,\underset{\alpha}{\tilde{\nu}} + \underset{\alpha}{\rho}\,\underset{\alpha}{\grave{k}}\,\underset{\alpha}{\grave{\nu}}\big)\big(\underset{\alpha}{\grave{\nu}}-\dot{\nu}\big)\Big] \\
& + \Big[\mathbf{q} - \Sigma\,\underset{\alpha}{\mathbf{T}}^{T}\underset{\alpha}{\mathbf{u}} - \Sigma\,\underset{\alpha}{\mathbf{h}}\big(\underset{\alpha}{\grave{\nu}}-\dot{\nu}\big)\Big]\cdot\operatorname{grad}\vartheta \\
& + \vartheta\rho\,\Sigma\big(\underset{\alpha}{\overset{+}{\mathbf{m}}} - \underset{\alpha}{\overset{+}{c}}\,\underset{\alpha}{\grave{x}}\big)\cdot\underset{\alpha}{\mathbf{u}} + \vartheta\rho\,\Sigma\big(\underset{\alpha}{\overset{+}{v}} - \underset{\alpha}{\overset{+}{c}}\,\underset{\alpha}{k}\,\underset{\alpha}{\grave{\nu}}\big)\big(\underset{\alpha}{\grave{\nu}}-\dot{\nu}\big) \\
& - \frac{\vartheta}{\rho}\operatorname{tr}\big(\mathbf{T} - \Sigma\,\underset{\alpha}{\mathbf{T}}\big)\big(\Sigma\,\underset{\alpha}{\rho}\,\underset{\alpha}{\mathbf{D}} + \Sigma\,\underset{\alpha}{\mathbf{u}}\otimes\operatorname{grad}\underset{\alpha}{\rho}\big) - \vartheta\,\Sigma\operatorname{tr}\underset{\alpha}{\mathbf{T}}^{T}\underset{\alpha}{\mathbf{G}} \\
& - \frac{\vartheta}{\rho k}\big(\mathbf{h} - \Sigma\,\underset{\alpha}{\mathbf{h}}\big)\cdot\Big[\Sigma\,\underset{\alpha}{\rho}\,\underset{\alpha}{\grave{k}}\operatorname{grad}\underset{\alpha}{\grave{\nu}} + \Sigma\big(\underset{\alpha}{\grave{\nu}}-\dot{\nu}\big)\big(\operatorname{grad}\underset{\alpha}{\rho}\,\underset{\alpha}{k}\big)\Big] - \vartheta\Sigma\,\underset{\alpha}{\mathbf{h}}\cdot\operatorname{grad}\underset{\alpha}{\grave{\nu}} \\
& - \vartheta\big(\rho\dot{k}\dot{\nu}^{2} - \Sigma\,\underset{\alpha}{\rho}\,\underset{\alpha}{g}\,\underset{\alpha}{\grave{\nu}}\big)\,.
\end{aligned}
\tag{5.28}
$$

6. Constitutive Equations for a Special Two-Component Mixture

Although constitutive equations may be written for a general mixture and the restrictions imposed on them by the entropy inequality (5.15) may be found, the resulting algebraic manipulations are quite tedious. I consider here a special two-component mixture, one component being a granular material as defined by Goodman and Cowin [2], the other a fluid of complexity 1. In particular

$$\underset{1}{k} = 0, \quad \underset{2}{\grave{k}} = 0. \tag{6.1}$$

Define

$$\overline{L} = \{\rho_{\alpha}, \rho_{\alpha}, \text{grad}\, \rho_{\alpha}, \acute{x}_{\alpha}, \text{grad}\, \acute{x}_{\alpha}, \vartheta, \text{grad}\, \vartheta, k_{2}, \nu_{\alpha}, \text{grad}\, \nu_{\alpha}, \acute{\nu}_{\alpha}\}, \quad (6.2)$$

where $\underset{\alpha}{\rho_{o}}(\underset{\sim}{X}) = \rho_{\alpha}(\underset{\sim}{x}_{\alpha}, 0)$, and $\alpha = 1, 2$.

Assume that each of the quantities

$$\psi, \eta, \underset{\sim}{q}, \hat{\underset{\sim}{\Phi}}, \underset{\sim}{T}_{\alpha}, \overset{+}{\underset{\sim}{m}}_{\alpha} - \overset{+}{c}_{\alpha}\acute{\underset{\sim}{x}}_{\alpha}, \hat{c}_{\alpha}, \underset{\sim}{h}_{\sigma}, \overset{+}{v}_{a}, g_{\alpha} \quad (6.3)$$

depends on $\overline{L}$. The material defined by this constitutive assumption will be called a mixture of a granular medium with a fluid.

Constitutive equations are often assumed to be subject to certain restrictions, one of which is called the "principle of frame-indifference". A change of frame is defined by

$$\begin{aligned} \underset{\sim}{x}^{*} &= \underset{\sim}{x}_{o}^{*} + \underset{\sim}{Q}(\underset{\sim}{x} - \underset{\sim}{x}_{o}), \quad \underset{\sim}{Q}\underset{\sim}{Q}^{T} = \underset{\sim}{1}, \\ \rho_{\alpha}^{*} &= \rho_{\alpha}, \\ \nu_{\alpha}^{*} &= \nu_{\alpha}, \\ k_{\alpha}^{*} &= k_{\alpha}, \end{aligned} \quad (6.4)$$

This is a generalization of the usual definition.

Define

$$\overline{L}^{*} = \{\underset{\alpha}{\rho_{0}},\ \underset{\alpha}{\rho},\ \underset{\sim}{Q}\ \mathrm{grad}\ \underset{\alpha}{\rho},\ \underset{\alpha}{\grave{\underset{\sim}{x}}}^{*} + Q(\underset{\sim}{x}-\underset{\sim}{x}_{0}) + \underset{\sim}{Q}\underset{\alpha}{\grave{\underset{\sim}{x}}},\ \underset{\sim}{Q}\ \mathrm{grad}\,\underset{\alpha}{\grave{\underset{\sim}{x}}}\,\underset{\sim}{Q}^{T} + \dot{\underset{\sim}{Q}}\underset{\sim}{Q}^{T},\ \vartheta,$$

$$\underset{\sim}{Q}\ \mathrm{grad}\ \vartheta,\ \underset{2}{k},\ \underset{\alpha}{\nu},\ \underset{\sim}{Q}\ \mathrm{grad}\ \underset{\alpha}{\nu},\ \underset{\alpha}{\grave{\nu}}\}\ .$$

Consider scalar-, vector-, and tensor-valued functions $s(\overline{L})$, $\underset{\sim}{w}(\overline{L})$, $\underset{\alpha}{\underset{\sim}{T}}(\overline{L})$. These functions are said to be frame-indifferent if

$$s(\overline{L}^{*}) = s(\overline{L})\ ,$$

$$\underset{\sim}{w}(\overline{L}^{*}) = \underset{\sim}{Q}\underset{\sim}{w}(\overline{L}), \tag{6.5}$$

$$\underset{\alpha}{\underset{\sim}{T}}(\overline{L}^{*}) = \underset{\sim}{Q}\underset{\alpha}{\underset{\sim}{T}}(\overline{L})\underset{\sim}{Q}^{T}\ .$$

I postulate that each of the quantities in (6.3) is frame-indifferent. Then by a familiar argument, dependence of these functions on $\overline{L}$ reduces to dependence on L, where

$$L = \{\underset{\alpha}{\rho_{0}},\ \underset{\alpha}{\rho},\ \mathrm{grad}\,\underset{\alpha}{\rho},\ \underset{\sim}{v},\ \underset{\alpha}{\underset{\sim}{D}},\ \underset{\sim}{\Omega},\ \vartheta,\ \mathrm{grad}\ \vartheta,\ \underset{2}{k},\ \underset{\alpha}{\nu},\ \mathrm{grad}\ \underset{\alpha}{\nu},\ \underset{\alpha}{\grave{\nu}}\}, \tag{6.6}$$

the dependence on vectors is only through their inner products, and

$$\underset{\sim}{\Omega} = \underset{1}{\underset{\sim}{W}} - \underset{2}{\underset{\sim}{W}}, \tag{6.7}$$

$$\underset{\sim}{v} = \underset{1}{\grave{\underset{\sim}{x}}} - \underset{2}{\grave{\underset{\sim}{x}}} \equiv \underset{1}{\underset{\sim}{u}} - \underset{2}{\underset{\sim}{u}}\ . \tag{6.8}$$

Furthermore, representation theorems for (6.3) as functions of L are known, although quite complicated. In the special case where the dependence on the vector and tensor variables in L is linear,

$$s(L) = \bar{s}, \tag{6.9}$$

$$\underset{\sim}{w}(L) = \Sigma \underset{a}{w_\rho}\, \mathrm{grad}\, \underset{a}{\rho} + w_D \underset{\sim}{v} + w_T\, \mathrm{grad}\, \vartheta + \Sigma \underset{a}{w_\nu}\, \mathrm{grad}\, \underset{a}{\nu}\,, \tag{6.10}$$

$$\overset{s}{\underset{a}{\underset{\sim}{T}}}(L) = -\underset{a}{p}\underset{\sim}{1} + \Sigma \underset{ab}{\xi}\, (\mathrm{tr}\, \underset{b}{\underset{\sim}{D}})\underset{\sim}{1} + 2\Sigma \underset{ab}{\eta}\, \underset{b}{\underset{\sim}{D}}\,, \tag{6.11}$$

$$\overset{a}{\underset{a}{\underset{\sim}{T}}}(L) = -\mu\underset{\sim}{\Omega}\,, \tag{6.12}$$

where $\overset{s}{\underset{a}{\underset{\sim}{T}}}$ and $\overset{a}{\underset{a}{\underset{\sim}{T}}}$ are the symmetric and skew parts of $\underset{a}{\underset{\sim}{T}}$, the summations are over the two components, and the functions $\bar{s}$, $\underset{a}{w_\rho}$, w_D, w_T, $\underset{a}{w_\nu}$, $\underset{a}{p}$, $\underset{ab}{\xi}$, $\underset{ab}{\eta}$, μ are functions of ℓ, where

$$\ell = \{\underset{a}{\rho_o}, \underset{a}{\rho}, \vartheta, \underset{2}{k}, \underset{a}{\nu}, \underset{a}{\grave{\nu}}\}\,. \tag{6.13}$$

I impose the restriction that L and ℓ do not in fact contain $\underset{1}{\rho_o}$, so that the first constituent is in fact a fluid. In addition, I assume that

$$\overset{+}{\underset{a}{c}} = 0, \tag{6.14}$$

thus excluding certain physical phenomena, including chemical reactions.

By (2.19), $(2.22)_3$, (5.19), (6.7) and some straightforward algebra, it may be shown that

$$\Sigma\, \mathrm{tr}(\mathbf{T}_\alpha^T \mathbf{G}_\alpha) = \mathrm{tr}(\overset{s}{\mathbf{T}}_1 \mathbf{D}_1 + \overset{s}{\mathbf{T}}_2 \mathbf{D}_2 + \overset{a}{\mathbf{T}}_1 \boldsymbol{\Omega}^T)\ . \tag{6.15}$$

Also, by $(2.23)_2$ and (6.8)

$$\Sigma\, \overset{+}{\mathbf{m}}_\alpha \cdot \mathbf{u}_\alpha = \overset{+}{\mathbf{m}}_1 \cdot \mathbf{v}\ , \tag{6.16}$$

and by $(2.23)_5$

$$\Sigma\, \overset{+}{v}_\alpha (\grave{\nu}_\alpha - \dot{\nu}) = \overset{+}{v}_1 (\grave{\nu}_1 - \grave{\nu}_2)\ . \tag{6.17}$$

By (1.6), (1.10), (2.11) and (6.8),

$$\Sigma\, \rho_\alpha \grave{\dot{\mathbf{x}}}_\alpha \cdot \mathbf{u}_\alpha = \frac{\rho_1 \rho_2}{\rho}\, \mathbf{v} \cdot \dot{\mathbf{v}}\ . \tag{6.18}$$

It is easily shown that

$$\overset{\frown}{\mathbf{x}}_\alpha = \grave{\mathbf{x}}_\alpha + \mathbf{G}_\alpha \mathbf{u}_\alpha\ , \tag{6.19}$$

so

$$\begin{aligned} \Sigma\, \rho_\alpha \overset{\frown}{\mathbf{x}}_\alpha \cdot \mathbf{u}_\alpha &= \Sigma\, \rho_\alpha \grave{\dot{\mathbf{x}}}_\alpha \cdot \mathbf{u}_\alpha + \Sigma\, \rho_\alpha \mathbf{u}_\alpha \cdot \mathbf{G}_\alpha \mathbf{u}_\alpha\ , \\ &= \Sigma\, \rho_\alpha \grave{\dot{\mathbf{x}}}_\alpha \cdot \mathbf{u}_\alpha + \Sigma\, \rho_\alpha \mathbf{u}_\alpha \cdot \mathbf{D}_\alpha \mathbf{u}_\alpha\ , \end{aligned} \tag{6.20}$$

or, by (6.18)

$$\Sigma\, \rho_\alpha \overset{\frown}{\mathbf{x}}_\alpha \cdot \mathbf{u}_\alpha = \frac{\rho_1 \rho_2}{\rho}\, \mathbf{v} \cdot \dot{\mathbf{v}} + \Sigma\, \rho_\alpha \mathbf{u}_\alpha \cdot \mathbf{D}_\alpha \mathbf{u}_\alpha\ . \tag{6.21}$$

By a set of steps similar to those leading to (6.21), it may be shown that, for a two-constituent mixture of granular materials

$$\sum (\rho_\alpha k_\alpha \tilde{\nu}_\alpha + \rho_\alpha \grave{k}_\alpha \grave{\nu}_\alpha)(\grave{\nu}_\alpha - \dot{\nu})$$
$$= [\frac{\rho_1\rho_2 k_1 k_2}{\rho k}(\tilde{\nu}_1 - \tilde{\nu}_2) + \frac{\rho_1\rho_2}{\rho k}(\grave{k}_1 k_2 \grave{\nu}_1 - k_1 \grave{k}_2 \grave{\nu}_2](\grave{\nu}_1 - \grave{\nu}_2) \,. \tag{6.22}$$

However, in the special case $k_1 = 0$, by (1.11)

$$\dot{\nu} = \grave{\nu}_2 \,, \tag{6.23}$$

and

$$\sum (\rho_\alpha k_\alpha \tilde{\nu}_\alpha + \rho_\alpha \grave{k}_\alpha \grave{\nu}_\alpha)(\grave{\nu}_\alpha - \dot{\nu}) = 0 \,. \tag{6.24}$$

By (6.14) - (6.17), (6.21) and (6.24), (5.28) becomes

$$\begin{aligned}
\rho\eta\frac{\dot{\vartheta}}{\vartheta} - \rho\vartheta\dot{\psi} \geq {} & \operatorname{div}\hat{\boldsymbol{\phi}} - \vartheta[\frac{\rho_1\rho_2}{\rho}\,\mathbf{v}\cdot\dot{\mathbf{v}} + \Sigma\rho_\alpha\mathbf{u}_\alpha\cdot\mathbf{D}_\alpha\mathbf{u}_\alpha] \\
& + [\mathbf{q} - \Sigma\mathbf{T}_\alpha^T\mathbf{u}_\alpha - \mathbf{h}_1(\grave{\nu}_1 - \dot{\nu})]\cdot\operatorname{grad}\vartheta + \vartheta\rho\overset{+}{\mathbf{m}}_1\cdot\mathbf{v} + \vartheta\rho\overset{+}{v}_1(\grave{\nu}_1 - \dot{\nu}) \\
& - \frac{\vartheta}{\rho}\operatorname{tr}(\mathbf{T} - \Sigma\mathbf{T}_\alpha)(\Sigma\rho_\alpha\mathbf{D}_\alpha + \Sigma\mathbf{u}_\alpha\otimes\operatorname{grad}\rho_\alpha) \\
& - \vartheta\operatorname{tr}(\overset{s}{\mathbf{T}}_1\mathbf{D}_1 + \overset{s}{\mathbf{T}}_2\mathbf{D}_2 + \overset{a}{\mathbf{T}}_1\boldsymbol{\Omega}^T) \\
& - \vartheta(\mathbf{h} - \Sigma\mathbf{h}_\alpha)\cdot\operatorname{grad}\grave{\nu}_2 - \vartheta\Sigma\mathbf{h}_\alpha\cdot\operatorname{grad}\grave{\nu}_\alpha \\
& - \vartheta(\rho\dot{k}\dot{\nu}^2 - \Sigma\rho_\alpha g_\alpha\grave{\nu}_\alpha) \,.
\end{aligned} \tag{6.25}$$

A result derived in a fashion similar to (6.19) is

$$\grave{\rho}_\alpha = \dot{\rho}_\alpha + \underset{\sim}{u}_\alpha \cdot \operatorname{grad} \rho_\alpha \,. \tag{6.26}$$

By the continuity equation $(2.7)_1$ and (6.14), it follows from (6.26) that

$$\dot{\rho}_\alpha = -\underset{\sim}{u}_\alpha \cdot \operatorname{grad} \rho_\alpha - \rho_\alpha \operatorname{tr} \underset{\sim}{D}_\alpha \,. \tag{6.27}$$

From a familiar commutation formula,

$$\frac{d}{dt}(\operatorname{grad} \rho_\alpha) = \operatorname{grad} \dot{\rho}_\alpha - \underset{\sim}{G}^T \operatorname{grad} \rho_\alpha \,. \tag{6.28}$$

Taking the gradient of (6.27) and using (5.20) and (6.28), yields

$$\begin{aligned} \frac{d}{dt}(\operatorname{grad} \rho_\alpha) = {} & +\underset{\sim}{G}_\alpha^T \operatorname{grad} \rho_\alpha - (\operatorname{grad} \operatorname{grad} \rho_\alpha)\underset{\sim}{u}_\alpha \\ & - (\operatorname{tr} \underset{\sim}{D}_\alpha)\operatorname{grad} \rho_\alpha - \rho_\alpha \operatorname{grad}(\operatorname{tr} \underset{\sim}{D}_\alpha) \,. \end{aligned} \tag{6.29}$$

I note the following result:

$$\dot{\nu}_\alpha = \grave{\nu}_\alpha - \underset{\sim}{u}_\alpha \cdot \operatorname{grad} \nu_\alpha \,. \tag{6.30}$$

By (6.27), (6.29) and (6.30) then

$$
\begin{aligned}
\dot{\psi} = &-\sum_{\alpha} \frac{\partial\psi}{\partial\rho_{\alpha}} \left(\underset{\sim}{u}_{\alpha} \cdot \operatorname{grad} \rho_{\alpha} + \rho_{\alpha} \operatorname{tr} \underset{\sim}{D}_{\alpha}\right) \\
&- \sum_{\alpha} \frac{\partial\psi}{\partial(\operatorname{grad} \rho_{\alpha})} \cdot \Big[\underset{\sim}{G}^{T} \operatorname{grad} \rho_{\alpha} + (\operatorname{grad} \operatorname{grad} \rho_{\alpha}) \underset{\sim}{u}_{\alpha} \\
&\qquad + (\operatorname{tr} \underset{\sim}{D}_{\alpha}) \operatorname{grad} \rho_{\alpha} + \rho_{\alpha} \operatorname{grad}(\operatorname{tr} \underset{\sim}{D}_{\alpha})\Big] \\
&+ \frac{\partial\psi}{\partial\underset{\sim}{v}} \cdot \dot{\underset{\sim}{v}} + \sum_{\alpha} \operatorname{tr} \frac{\partial\psi}{\partial\underset{\sim}{D}_{\alpha}} \dot{\underset{\sim}{D}}_{\alpha} + \operatorname{tr} \frac{\partial\psi}{\partial\underset{\sim}{\Omega}} \dot{\underset{\sim}{\Omega}} + \frac{\partial\psi}{\partial\vartheta} \dot{\vartheta} \\
&+ \frac{\partial\psi}{\partial(\operatorname{grad} \vartheta)} \cdot \frac{d}{dt} \operatorname{grad} \vartheta + \frac{\partial\psi}{\partial k_{2}} \dot{k}_{2} \\
&+ \sum_{\alpha} \frac{\partial\psi}{\partial\nu_{\alpha}} \left(\grave{\nu}_{\alpha} - \underset{\sim}{u}_{\alpha} \cdot \operatorname{grad} \nu_{\alpha}\right) \\
&+ \sum_{\alpha} \frac{\partial\psi}{\partial(\operatorname{grad} \nu_{\alpha})} \cdot \frac{d}{dt} (\operatorname{grad} \nu_{\alpha}) + \sum_{\alpha} \frac{\partial\psi}{\partial\grave{\nu}_{\alpha}} \cdot \frac{d}{dt} (\operatorname{grad} \grave{\nu}_{\alpha}) .
\end{aligned}
\tag{6.31}
$$

It is obvious that the computation of div $\hat{\underset{\sim}{\phi}}$ will involve gradients of second-order tensors, which are third-order tensors. Such quantities are in general ill-adapted to the system of notation used here. I choose a system of orthogonal Cartesian coordinates, and establish the notation used in terms of components referred to these coordinates.

Let $\underset{\sim}{a}$ be a vector and $\underset{\sim}{\mathit{a}}$ be a tensor. Read the symbol "~" as "has Cartesian components", so that

$$
\underset{\sim}{a} \sim a_{i} ,
$$

$$
\underset{\sim}{\mathit{a}} \sim \mathit{a}_{ik} .
$$

Then by definition

$$\operatorname{grad} \underset{\sim}{a} \sim a_{jk,\ell} \, ,$$

where the comma indicates partial differentiation, and

$$\frac{\partial \hat{\underset{\sim}{\phi}}}{\partial \underset{\sim}{a}} \sim \frac{\partial \hat{\phi}_i}{\partial a_{jk}} \, .$$

Define

$$\operatorname{tr}\left(\frac{\partial \hat{\underset{\sim}{\phi}}}{\partial \underset{\sim}{a}} \operatorname{grad} \underset{\sim}{a}\right) \sim \frac{\partial \hat{\phi}_i}{\partial a_{jk}} a_{jk,i} \, .$$

Note the identity

$$\Omega_{ij,k} = (\underset{1}{D}_{ki,j} - \underset{1}{D}_{kj,i}) - (\underset{2}{D}_{ki,j} - \underset{2}{D}_{kj,i}), \tag{6.34}$$

which will be abbreviated as

$$\operatorname{grad} \underset{\sim}{\Omega} = (\operatorname{grad} \underset{1}{\underset{\sim}{D}})^{\times} - (\operatorname{grad} \underset{2}{\underset{\sim}{D}})^{\times} . \tag{6.35}$$

Differentiating (6.30), gives

$$\operatorname{grad} \underset{a}{\grave{\nu}} = \frac{d}{dt} \operatorname{grad} \underset{a}{\nu} + (\operatorname{grad} \operatorname{grad} \underset{a}{\nu}) \underset{a}{\underset{\sim}{u}} \tag{6.36}$$

Computing div $\hat{\underset{\sim}{\phi}}$ is now a straightforward task. By (6.35) and (6.36),

$$
\begin{aligned}
\operatorname{div} \hat{\underset{\sim}{\phi}} = {} & \frac{\partial \hat{\underset{\sim}{\phi}}}{\partial \underset{2}{\rho_o}} \cdot \operatorname{grad} \underset{2}{\rho_o} + \Sigma \frac{\partial \hat{\underset{\sim}{\phi}}}{\partial \underset{a}{\rho}} \cdot \text{gra[illegible]} \\
& + \Sigma \operatorname{tr} \frac{\partial \hat{\underset{\sim}{\phi}}}{\partial (\operatorname{grad} \underset{a}{\rho})} \operatorname{grad} \operatorname{grad} \underset{a}{\rho} + \text{tr[illegible]} \operatorname{grad} \underset{\sim}{v} \\
& + \Sigma \operatorname{tr} \frac{\partial \hat{\underset{\sim}{\phi}}}{\partial \underset{a}{\underset{\sim}{D}}} \operatorname{grad} \underset{a}{\underset{\sim}{D}} + \operatorname{tr} \frac{\partial \hat{\underset{\sim}{\phi}}}{\partial \underset{\sim}{\Omega}} (\operatorname{grad} \underset{1}{\underset{\sim}{D}})^{>} - \operatorname{tr} \frac{\partial \hat{\underset{\sim}{\phi}}}{\partial \underset{\sim}{\Omega}} (\operatorname{grad} \underset{2}{\underset{\sim}{D}})^{\times} \\
& + \frac{\partial \hat{\underset{\sim}{\phi}}}{\partial \vartheta} \cdot \operatorname{grad} \vartheta + \operatorname{tr} \frac{\partial \hat{\underset{\sim}{\phi}}}{\partial (\operatorname{grad} \vartheta)} \operatorname{grad} \operatorname{grad} \vartheta \\
& + \frac{\partial \hat{\underset{\sim}{\phi}}}{\partial \underset{2}{k}} \cdot \operatorname{grad} \underset{2}{k} + \Sigma \frac{\partial \hat{\underset{\sim}{\phi}}}{\partial \underset{a}{\nu}} \cdot \operatorname{grad} \underset{a}{\nu} + \Sigma \operatorname{tr} \frac{\partial \hat{\underset{\sim}{\phi}}}{\partial (\operatorname{grad} \underset{a}{\nu})} \operatorname{grad} \operatorname{grad} \underset{a}{\nu} \\
& + \Sigma \frac{\partial \hat{\underset{\sim}{\phi}}}{\partial \underset{a}{\grave{\underset{\sim}{\gamma}}}} \cdot \left[\frac{d}{dt} \operatorname{grad} \underset{a}{\nu} + (\operatorname{grad} \operatorname{grad} \underset{a}{\nu}) \underset{a}{\underset{\sim}{u}}\right] .
\end{aligned}
\tag{6.37}
$$

Note that, by $(2.12)_2$, (6.1) and (6.14)

$$
\underset{2}{\rho} \underset{2}{\acute{k}} = -\operatorname{div}(\underset{2}{\rho} \underset{2}{k} \underset{2}{\underset{\sim}{u}}), \tag{6.38}
$$

so that

$$
\rho \dot{\underset{2}{k}} = -\underset{2}{k} \operatorname{div}(\rho \underset{22}{\underset{\sim}{u}}) - \rho \underset{22}{\underset{\sim}{u}} \cdot \operatorname{grad} \underset{2}{k} . \tag{6.39}
$$

By (6.23) and (6.39),

$$
\rho \dot{k} \grave{\nu}^2 = -\underset{22}{k \grave{\nu}^2} \operatorname{div}(\underset{22}{\rho \underset{\sim}{u}}) - \underset{22}{\rho \grave{\nu}^2} \underset{2}{\underset{\sim}{u}} \cdot \operatorname{grad} \underset{2}{k} . \tag{6.40}
$$

Also

$$
\dot{\underset{2}{k}} = -\underset{2}{\underset{\sim}{u}} \cdot \operatorname{grad} \underset{2}{k} . \tag{6.41}
$$

Substituting (6.33), (6.40) and (6.41) into (6.25) then yields

$$
\begin{aligned}
&\rho\eta\frac{\dot{\vartheta}}{\vartheta} - \rho\vartheta\Big[-\Sigma\frac{\partial\psi}{\partial\rho_\alpha}(\mathbf{u}_\alpha\cdot\operatorname{grad}\rho_\alpha + \rho_\alpha\operatorname{tr}\mathbf{D}_\alpha) - \Sigma\frac{\partial\psi}{\partial(\operatorname{grad}\rho_\alpha)}\cdot[\mathbf{G}_\alpha^T\operatorname{grad}\rho_\alpha + (\operatorname{grad}\operatorname{grad}\rho_\alpha)\mathbf{u}_\alpha + (\operatorname{tr}\mathbf{D}_\alpha)\operatorname{grad}\rho_\alpha + \rho_\alpha\operatorname{grad}(\operatorname{tr}\mathbf{D}_\alpha)]\\
&+\frac{\partial\psi}{\partial\mathbf{v}}\cdot\dot{\mathbf{v}} + \Sigma\operatorname{tr}\frac{\partial\psi}{\partial\mathbf{D}_\alpha}\dot{\mathbf{D}}_\alpha + \operatorname{tr}\frac{\partial\psi}{\partial\boldsymbol{\Omega}}\dot{\boldsymbol{\Omega}} + \frac{\partial\psi}{\partial\vartheta}\dot{\vartheta} + \frac{\partial\psi}{\partial(\operatorname{grad}\vartheta)}\cdot\frac{d}{dt}(\operatorname{grad}\vartheta) - \frac{\partial\psi}{\partial k_2}\mathbf{u}_2\cdot\operatorname{grad}k_2\\
&+\Sigma\frac{\partial\psi}{\partial\nu_\alpha}(\grave{\nu}_\alpha - \mathbf{u}_\alpha\cdot\operatorname{grad}\nu_\alpha) + \Sigma\frac{\partial\psi}{\partial(\operatorname{grad}\nu_\alpha)}\cdot\frac{d}{dt}\operatorname{grad}\nu_\alpha + \Sigma\frac{\partial\psi}{\partial\grave{\nu}_\alpha}\frac{d\grave{\nu}_\alpha}{dt}\Big] \geq\\
&\quad\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\rho_{2_o}}\cdot\operatorname{grad}\rho_{2_o} + \Sigma\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\rho_\alpha}\cdot\operatorname{grad}\rho_\alpha + \Sigma\operatorname{tr}\frac{\partial\hat{\boldsymbol{\phi}}}{\partial(\operatorname{grad}\rho_\alpha)}\operatorname{grad}\operatorname{grad}\rho_\alpha + \operatorname{tr}\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\mathbf{v}}\operatorname{grad}\mathbf{v}\\
&+\Sigma\operatorname{tr}\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\mathbf{D}_\alpha}\operatorname{grad}\mathbf{D}_\alpha + \operatorname{tr}\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\boldsymbol{\Omega}}(\operatorname{grad}\mathbf{D}_1)^\times - \operatorname{tr}\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\boldsymbol{\Omega}}(\operatorname{grad}\mathbf{D}_2)^\times + \frac{\partial\hat{\boldsymbol{\phi}}}{\partial\vartheta}\cdot\operatorname{grad}\vartheta + \operatorname{tr}\frac{\partial\hat{\boldsymbol{\phi}}}{\partial(\operatorname{grad}\vartheta)}\operatorname{grad}\operatorname{grad}\vartheta\\
&+\frac{\partial\hat{\boldsymbol{\phi}}}{\partial k_2}\cdot\operatorname{grad}k_2 + \Sigma\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\nu_\alpha}\cdot\operatorname{grad}\nu_\alpha + \Sigma\operatorname{tr}\frac{\partial\hat{\boldsymbol{\phi}}}{\partial(\operatorname{grad}\nu_\alpha)}\operatorname{grad}\operatorname{grad}\nu_\alpha + \Sigma\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\grave{\nu}_\alpha}\cdot[\frac{d}{dt}\operatorname{grad}\nu_\alpha + (\operatorname{grad}\operatorname{grad}\nu_\alpha)\mathbf{u}_\alpha] \qquad (6.42)\\
&-\vartheta[\frac{\rho_1\rho_2}{\rho}\mathbf{v}\cdot\dot{\mathbf{v}} + \Sigma\rho_\alpha\mathbf{u}_\alpha\cdot\mathbf{D}_\alpha\mathbf{u}_\alpha]\\
&+[\mathbf{q} - \Sigma\mathbf{T}_\alpha^T\mathbf{u}_\alpha - \mathbf{h}_1(\grave{\nu}_1 - \dot{\nu})]\cdot\operatorname{grad}\vartheta + \vartheta\rho\overset{+}{\mathbf{m}}_1\cdot\mathbf{v} + \vartheta\rho\overset{+}{\nu}_1(\grave{\nu}_1 - \dot{\nu})\\
&-\frac{\vartheta}{\rho}\operatorname{tr}(\mathbf{T} - \Sigma\mathbf{T}_\alpha)(\Sigma\rho_\alpha\mathbf{D}_\alpha + \Sigma\mathbf{u}_\alpha\otimes\operatorname{grad}\rho_\alpha) - \vartheta\operatorname{tr}(\overset{s}{\mathbf{T}}_1\mathbf{D}_1 + \overset{s}{\mathbf{T}}_2\mathbf{D}_2 + \overset{a}{\mathbf{T}}_1\boldsymbol{\Omega}^T)\\
&-\vartheta(\mathbf{h} - \Sigma\mathbf{h}_\alpha)\cdot[\frac{d}{dt}\operatorname{grad}\nu_2 + (\operatorname{grad}\operatorname{grad}\nu_2)\mathbf{u}_2] - \vartheta\Sigma\mathbf{h}_\alpha\cdot[\frac{d}{dt}\operatorname{grad}\nu_\alpha + (\operatorname{grad}\nu_\alpha + (\operatorname{grad}\operatorname{grad}\nu_\alpha)\mathbf{u}_\alpha] +\\
&\qquad+\vartheta\rho_2\grave{\nu}_2^2\mathbf{u}_2\cdot\operatorname{grad}k_2 + \vartheta k_2\grave{\nu}_2^2\operatorname{div}(\rho_2\mathbf{u}_2) + \vartheta\Sigma\rho_\alpha g_\alpha\grave{\nu}_\alpha.
\end{aligned}
$$

A thermodynamic process is defined in a fashion similar to that for a conventional continuum. Each balance law with the exception of the one for mass contains one term which may be adjusted arbitrarily, and there exists at least one thermodynamic process in which $\dot{\underset{\sim}{v}}$, $\underset{\alpha}{\dot{\underset{\sim}{D}}}$, grad $\underset{\alpha}{\underset{\sim}{D}}$, $\dot{\underset{\sim}{\Omega}}$, grad $\underset{\alpha}{\rho_O}$, grad grad $\underset{\alpha}{\rho}$, $\dot{\vartheta}$, $\frac{d}{dt}$grad ϑ, grad grad ϑ, $\frac{d}{dt}\underset{\alpha}{\grave{\nu}}$, $\frac{d}{dt}$grad $\underset{\alpha}{\nu}$, grad grad $\underset{\alpha}{\nu}$ and grad $\underset{2}{k}$ may be chosen arbitrarily and independently of any other term in the inequality. This implies the following results*:

a. ψ is independent of $\underset{\alpha}{\underset{\sim}{D}}$, $\underset{\sim}{\Omega}$, grad ϑ, $\underset{\alpha}{\grave{\nu}}$,

b. $\frac{\partial\psi}{\partial\underset{\sim}{v}} = \frac{\rho_1\rho_2}{\rho^2}\underset{\sim}{v}$, (6.43)

so that

$$\psi = \psi_I(\underset{2}{\rho_O}, \underset{\alpha}{\rho}, \text{grad}\, \underset{\alpha}{\rho}, \vartheta, \underset{2}{k}, \underset{\alpha}{\nu}, \text{grad}\, \underset{\alpha}{\nu}) + \frac{1}{2}\Sigma\frac{\underset{\alpha}{\rho}}{\rho}\underset{\alpha}{\underset{\sim}{u}}\cdot\underset{\alpha}{\underset{\sim}{u}}\,. \qquad (6.44)$$

Furthermore, ψ_I depends on grad $\underset{\alpha}{\rho}$ and grad $\underset{\alpha}{\nu}$ only through their inner products grad $\underset{\alpha}{\rho}$ · grad $\underset{b}{\rho}$, grad $\underset{\alpha}{\rho}$ · grad $\underset{b}{\nu}$, grad $\underset{\alpha}{\nu}$ · grad $\underset{b}{\nu}$.

c. $\frac{\partial\psi}{\partial\vartheta} = \frac{\eta}{\vartheta^2}$. (6.45)

d. $\frac{\partial\hat{\phi}}{\partial\underset{2}{k}} = \rho\vartheta\underset{2}{\underset{\sim}{u}}(\frac{\partial\psi}{\partial\underset{2}{k}} - \underset{2}{\grave{\nu}}^2)$. (6.46)

e. $(\frac{\partial\hat{\phi}}{\partial(\text{grad}\,\vartheta)})^s = \underset{\sim}{0}$, (6.47)

---

*These results reduce to those of Müller [3] in the case where both continua are ordinary fluids. I have corrected a minor printer's error in result b.

where, as before, the superscript ^{s} denotes the symmetric part of the indicated tensor.

f. $$\left(\frac{\partial}{\partial(\text{grad}\,\underset{\alpha}{\rho})}\left[\hat{\underset{\sim}{\Phi}} - \rho\vartheta\psi_I \underset{\sim}{u}_{\alpha}\right]\right)^s = \underset{\sim}{0}. \tag{6.48}$$

Here (6.44) has been used.

g. Again, I revert to Cartesian components.

$$\left(\rho\underset{1}{\rho}\vartheta \frac{\partial\psi_I}{\partial \underset{1}{\rho}_{,k}}\delta_{ij} - \frac{\partial\hat{\phi}_j}{\partial\Omega_{ki}} + \frac{\partial\hat{\phi}_k}{\partial\Omega_{ji}} - \frac{\partial\hat{\phi}_k}{\partial \underset{1}{D}_{ij}}\right)^{s_{ij}} = 0, \tag{6.49}$$

$$\left(\rho\underset{2}{\rho}\vartheta \frac{\partial\psi_I}{\partial \underset{2}{\rho}_{,k}}\delta_{ij} + \frac{\partial\hat{\phi}_j}{\partial\Omega_{ki}} - \frac{\partial\hat{\phi}_k}{\partial\Omega_{ji}} - \frac{\partial\hat{\phi}_k}{\partial \underset{2}{D}_{ij}}\right)^{s_{ij}} = 0, \tag{6.50}$$

where the superscript $^{s_{ij}}$ denotes the symmetric part with respect to i and j.

h. $$\vartheta\underset{1}{\underset{\sim}{h}} = \frac{\partial\hat{\underset{\sim}{\Phi}}}{\partial\underset{1}{\dot{\nu}}} + \vartheta\rho\frac{\partial\psi_I}{\partial(\text{grad}\,\underset{1}{\nu})}, \tag{6.51}$$

$$\vartheta\underset{2}{\underset{\sim}{h}} = \frac{\partial\hat{\underset{\sim}{\Phi}}}{\partial\dot{\nu}} + \vartheta\rho\frac{\partial\psi_I}{\partial(\text{grad}\,\underset{2}{\nu})} + \vartheta\rho \underset{2222}{k\grave{\nu}\underset{\sim}{u}}. \tag{6.52}$$

i. $$\left(\frac{\partial}{\partial(\text{grad}\,\underset{\alpha}{\nu})}\left[\hat{\underset{\sim}{\Phi}} - \rho\vartheta\psi_I \underset{\sim}{u}_{\alpha}\right]\right)^s = 0, \tag{6.53}$$

where in deriving (6.53), (6.51) and (6.52) have been used.

j. $\hat{\underset{\sim}{\Phi}}$ is independent of grad $\underset{2}{\hat{\rho}}_o$.

I note that

$$\operatorname{grad} \underset{\sim}{v} = \underset{\sim}{D}_1 - \underset{\sim}{D}_2 + \underset{\sim}{\Omega} \,. \tag{6.54}$$

Furthermore

$$\begin{aligned} \operatorname{div} \rho_2 \underset{\sim}{u}_2 &= \rho_2 \operatorname{div}(\grave{\underset{\sim}{x}}_2 - \dot{\underset{\sim}{x}}) + \underset{\sim}{u}_2 \cdot \operatorname{grad} \rho_2 \,, \\ &= \rho_2 \operatorname{tr} \underset{\sim}{D}_2 - \rho_2 \operatorname{tr} \underset{\sim}{D} + \underset{\sim}{u}_2 \cdot \operatorname{grad} \rho_2 \,. \end{aligned} \tag{6.55}$$

However, by (5.22)

$$\rho \operatorname{tr} \underset{\sim}{D} = \rho_1 \operatorname{tr} \underset{\sim}{D}_1 + \rho_2 \operatorname{tr} \underset{\sim}{D}_2 + \underset{\sim}{u}_1 \cdot \operatorname{grad} \rho_1 + \underset{\sim}{u}_2 \cdot \operatorname{grad} \rho_2 \,. \tag{6.56}$$

Substituting (6.56) into (6.55) and taking (1.6) into account, I obtain

$$\begin{aligned} \operatorname{div} \rho_2 \underset{\sim}{u}_2 = \frac{\rho_1 \rho_2}{\rho} (\operatorname{tr} \underset{\sim}{D}_2 - \operatorname{tr} \underset{\sim}{D}_1) &- \frac{\rho_2}{\rho} \underset{\sim}{u}_1 \cdot \operatorname{grad} \rho_1 \\ &+ \frac{\rho_1}{\rho} \underset{\sim}{u}_2 \cdot \operatorname{grad} \rho_2 \,. \end{aligned} \tag{6.57}$$

Substitution of (6.54), and (6.57) into (6.42) yields the residual inequality

$$
\begin{aligned}
&\rho\vartheta\frac{\partial\psi}{\partial\rho_1}\,\underset{\sim}{u}_1\cdot\operatorname{grad}\rho_1-\frac{\partial\hat{\phi}}{\partial\rho_1}\cdot\operatorname{grad}\rho_1+\frac{\vartheta}{\rho}\operatorname{tr}(\underset{\sim}{T}-\Sigma\underset{\sim}{T}_\alpha)\underset{\sim}{u}_1\otimes\operatorname{grad}\rho_1-\\
&\quad-\vartheta k\grave{\nu}^2_{22}\frac{\rho_2}{\rho}\,\underset{\sim}{u}_1\cdot\operatorname{grad}\rho_1+\\
&+\rho\vartheta\frac{\partial\psi}{\partial\rho_2}\,\underset{\sim}{u}_2\cdot\operatorname{grad}\rho_2-\frac{\partial\hat{\phi}}{\partial\rho_2}\cdot\operatorname{grad}\rho_2+\frac{\vartheta}{\rho}\operatorname{tr}(\underset{\sim}{T}-\Sigma\underset{\sim}{T}_\alpha)\underset{\sim}{u}_2\otimes\operatorname{grad}\rho_2+\\
&\quad+\vartheta k\grave{\nu}^2_{22}\frac{\rho_1}{\rho}\,\underset{\sim}{u}_2\cdot\operatorname{grad}\rho_2+\\
&+\rho\rho_1\frac{\vartheta\psi}{\partial\rho_1}\operatorname{tr}\underset{\sim}{D}_1+\rho\vartheta\frac{\partial\psi}{\partial(\operatorname{grad}\rho_1)}\cdot\underset{\sim}{D}_1\operatorname{grad}\rho_1+\rho\vartheta\frac{\partial\psi}{\partial(\operatorname{grad}\rho_1)}\cdot\operatorname{grad}\rho_1(\operatorname{tr}\underset{\sim}{D}_1)-\\
&\quad-\operatorname{tr}\frac{\partial\hat{\phi}}{\partial\underset{\sim}{v}}\cdot\underset{\sim}{D}_1+\vartheta\rho_1\underset{\sim}{u}_1\cdot\underset{\sim}{D}_1\underset{\sim}{u}_1+\vartheta\frac{\rho_1}{\rho}\operatorname{tr}(\underset{\sim}{T}-\Sigma\underset{\sim}{T}_\alpha)\underset{\sim}{D}_1+\vartheta\operatorname{tr}\overset{s}{\underset{\sim}{T}}_1\underset{\sim}{D}_1-\\
&\quad-\vartheta k\grave{\nu}^2_{22}\frac{\rho_1\rho_2}{\rho}\operatorname{tr}\underset{\sim}{D}_1+ \qquad\qquad (6.58)\\
&+\rho\rho_2\vartheta\frac{\partial\psi}{\partial\rho_2}\operatorname{tr}\underset{\sim}{D}_2+\rho\vartheta\frac{\partial\psi}{\partial(\operatorname{grad}\rho_2)}\cdot\underset{\sim}{D}_2\operatorname{grad}\rho_2+\rho\vartheta\frac{\partial\psi}{\partial(\operatorname{grad}\rho_2)}\cdot\operatorname{grad}\rho_2(\operatorname{tr}\underset{\sim}{D}_2)+\\
&\quad+\operatorname{tr}\frac{\partial\hat{\phi}}{\partial\underset{\sim}{v}}\underset{\sim}{D}_2+\vartheta\rho_2\underset{\sim}{u}_2\cdot\underset{\sim}{D}_2\underset{\sim}{u}_2+\vartheta\frac{\rho_2}{\rho}\operatorname{tr}(\underset{\sim}{T}-\Sigma\underset{\sim}{T}_\alpha)\underset{\sim}{D}_2+\vartheta\operatorname{tr}\overset{s}{\underset{\sim}{T}}_2\underset{\sim}{D}_2+\vartheta k\grave{\nu}_{22}\frac{\rho_1\rho_2}{\rho}\operatorname{tr}\underset{\sim}{D}_2-\\
&-\operatorname{tr}\Big[\Big(\frac{\partial\hat{\phi}}{\partial\underset{\sim}{v}}+\vartheta\overset{a}{\underset{\sim}{T}}_1\Big)\underset{\sim}{\Omega}\Big]+\Big[-\frac{\partial\hat{\phi}}{\partial\vartheta}-\underset{\sim}{g}+\Sigma\underset{\sim}{T}^T_\alpha\underset{\sim}{u}_\alpha+\underset{\sim}{h}_1(\grave{\nu}_1-\hat{\nu})\Big]\cdot\operatorname{grad}\vartheta-\\
&\quad-\vartheta\rho\overset{+}{\underset{\sim}{m}}_1\cdot\underset{\sim}{v}-\vartheta\rho\overset{+}{v}_1(\grave{\nu}_1-\hat{\nu})\\
&-\Sigma\Big(\frac{\partial\psi}{\partial\nu_\alpha}\underset{\sim}{u}_\alpha+\frac{\partial\hat{\phi}}{\partial\nu_\alpha}\Big)\cdot\operatorname{grad}\nu_\alpha-\Sigma\Big(\rho\vartheta\frac{\partial\psi}{\partial\nu_\alpha}+\vartheta\rho_\alpha g_\alpha\Big)\grave{\nu}_\alpha\geq 0\,.
\end{aligned}
$$

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The consequences of results a. - f. may be investigated by standard methods and constitute generalizations of known results (see, e.g., [3]) for a mixture of two constituents to the case where one of the components is a granular material. In particular a. and b. indicate that the Helmholtz free energy, in addition to being independent of peculiar stretchings $\underset{\alpha}{\underset{\sim}{D}}$, the relative spin $\underset{\sim}{\Omega}$, and the coldness gradient, grad ϑ, is independent of the peculiar volume distribution velocities $\underset{\alpha}{\grave{\nu}}$. Furthermore its dependance on diffusion velocities is made explicit by (6.44). Result c. is a familiar relation from thermodynamics, but result d. is new. Results h. generalize analogous results of Goodman and Cowin indicating, as might be expected, a contribution to peculiar equilibrated stress due to diffusion. Result g. constitutes a set of restrictions on the forms of the entropy flux and Helmholtz free energy. The restrictions imposed by the familiar relations e. and f., and the new relation i. are easily made explicit by a method due to Müller. Solving these equations, I obtain, in Cartesian tensor notation,

$$\hat{\phi}_i - \rho\underset{2}{\psi}_I v_i = (\underset{1}{\Lambda}_{ijkl}\, \underset{1}{\rho}_{,j}\, \underset{1}{\nu}_{,k} + \underset{1}{\Lambda}_{ijk}\, \underset{1}{\rho}_{,j} + \underset{1}{\Gamma}_{ijk}\, \underset{1}{\nu}_{,j} + \underset{1}{\Lambda}_{ij})\vartheta_{,j}$$
$$+ \underset{1}{\Delta}_{ijk}\, \underset{1}{\rho}_{,j}\, \underset{1}{\nu}_{,k} + \underset{1}{\Lambda}_{ij}\, \underset{1}{\nu}_{,i} + \underset{1}{\Gamma}_{ij}\, \underset{1}{\rho}_{,j} + \underset{1}{\Lambda}_{i}\,, \tag{6.59}$$

$$\hat{\phi}_i + \vartheta \underset{1}{\rho}\psi_I v_i = (\underset{2}{\Lambda}_{ijkl}\, \underset{1}{\rho}_{,j}\, \underset{2}{\nu}_{,k} + \underset{2}{\Lambda}_{ijk}\, \underset{2}{\rho}_{,j} + \underset{2}{\Gamma}_{ijk}\, \underset{2}{\nu}_{,j} + \underset{2}{\Lambda}_{ij})\vartheta_{,j}$$

$$+ \underset{2}{\Delta}_{ijk}\, \underset{2}{\rho}_{,j}\, \underset{2}{\nu}_{,k} + \underset{2}{\Lambda}_{ij}\, \underset{2}{\nu}_{,j} + \underset{2}{\Gamma}_{ij}\, \underset{2}{\rho}_{,j} + \underset{2}{\Delta}_i \, . \tag{6.60}$$

Here the coefficients Λ, Γ, Δ are skew in all indices, are independent of grad ϑ, and a subscript 1 or 2 indicates independence of grad $\underset{1}{\rho}$ and grad $\underset{1}{\nu}$ or grad $\underset{2}{\rho}$ and grad $\underset{2}{\nu}$, respectively.

It follows that

$$\rho\vartheta\psi_I v_i = (\underset{2}{\Delta}_{ijk}\underset{2}{\rho}_{,j}\underset{2}{\nu}_{,k} + \underset{2}{\Lambda}_{ij}\underset{2}{\nu}_{,i} + \underset{2}{\Gamma}_{ij}\underset{2}{\rho}_{,j} + \underset{2}{\Lambda}_i)$$

$$- (\underset{1}{\Delta}_{ijk}\underset{1}{\rho}_{,j}\underset{1}{\nu}_{,k} + \underset{1}{\Lambda}_{ij}\underset{1}{\nu}_{,i} + \underset{1}{\Gamma}_{ij}\underset{1}{\rho}_{,j} + \underset{1}{\Lambda}_i) \, . \tag{6.61}$$

Equations (6.59) - (6.61) may be used to derive a more explicit form for $\hat{\underset{\sim}{\phi}}$ by straightforward addition and substitution. Furthermore, the functions Λ, Δ, Γ may be recognized as combinations of the derivatives of $\rho\vartheta\psi_I \underset{\sim}{v}$ with respect to grad $\underset{\alpha}{\rho}$, grad $\underset{\alpha}{\nu}$ and the values of these derivatives with one or more of the parameters vanishing. Finally, it may be recalled that ψ_I and $\hat{\underset{\sim}{\phi}}$ depend on grad $\underset{\alpha}{\rho}$, grad $\underset{\alpha}{\nu}$ and grad ϑ only through their inner products, thus yielding an alternative form for ψ_I. All of these results involve straightforward calculations and yield somewhat complicated results. Since they are not needed in their full generality in the following sections, they are not recorded here.

7. Equilibrium

Let

$$X_A = \{\mathrm{grad}\ \underset{\alpha}{\rho},\ \underset{\alpha}{\tilde{D}},\ \tilde{\Omega},\ \mathrm{grad}\ \vartheta,\ \tilde{v},\ \mathrm{grad}\ \underset{\alpha}{\nu},\ \underset{\alpha}{\grave{\nu}}\}\ . \tag{7.1}$$

If $X_A = 0$, the mixture is said to be in equilibrium. Let the entropy production σ be the left-hand side of (6.58), so that inequality is

$$\sigma \geq 0\ . \tag{7.2}$$

Then σ has a minimum at equilibrium. Necessary condtions for this are

$$\frac{\partial\sigma}{\partial X_A} = 0 \ \text{at}\ X_A = 0, \tag{7.3}$$

$$\left\| \frac{\partial^2\sigma}{\partial X_A \partial X_B} \right\| \ \text{is non-negative definite at}\ X_A = 0\ . \tag{7.4}$$

By $(2.25)_1$ and (6.59) - (6.61), consequences of (7.3) are

$$\frac{\partial\hat{\Phi}}{\partial \underset{\alpha}{\rho}} = 0, \tag{7.5}$$

$$\vartheta \underset{1}{\overset{s}{\tilde{T}}} = \vartheta\rho \underset{1}{\rho} \frac{\partial\psi}{\partial \underset{1}{\rho}} \tilde{1} - \frac{\partial\hat{\tilde{\Phi}}}{\partial\tilde{v}} \equiv -\vartheta \underset{1}{p} \tilde{1}\ , \tag{7.6}$$

$$\vartheta \underset{2}{\overset{s}{\tilde{T}}} = \vartheta\rho \underset{2}{\rho} \frac{\partial\psi}{\partial \underset{2}{\rho}} \tilde{1} + \frac{\partial\hat{\tilde{\Phi}}}{\partial\tilde{v}} \equiv -\vartheta \underset{2}{p} \tilde{1}\ , \tag{7.7}$$

$$\underset{1}{\overset{a}{\tilde{T}}} = \tilde{0}\ , \tag{7.8}$$

$$\tilde{g} = \tilde{0}\ , \tag{7.9}$$

$$\underset{1}{\overset{+}{\underset{\sim}{m}}} = \underset{\sim}{0} \,, \tag{7.10}$$

$$\frac{\partial \hat{\underset{\sim}{\Phi}}}{\partial \underset{\alpha}{\nu}} = 0, \tag{7.11}$$

$$\rho \underset{11}{g} = -\rho \frac{\partial \psi}{\partial \underset{1}{\nu}} - \rho \underset{1}{\overset{+}{v}} \,, \tag{7.12}$$

$$\rho \underset{22}{g} = -\rho \frac{\partial \psi}{\partial \underset{2}{\nu}} + \rho \underset{1}{\overset{+}{v}} \,, \tag{7.13}$$

where all quantities are evaluated at equilibrium.

Equations (7.5) - (7.10) are familiar results from the thermostatics of a mixture of two ordinary fluids, and have been commented upon extensively by Müller [3]. Unlike the single component granular material of Goodman and Cowin, a mixture of a granular material and a fluid is incapable of sustaining a shear stress in equilibrium. The results (7.12) and (7.13) are analogous to those obtained in elasticity theory, stating that the intrinsic body forces are derived from the Helmholtz free energy and the growth of equilibrated force.

8. A Linear Theory

Consider an expansion about equilibrium, linear in X_A as defined by (7.1) of the quantities ψ, $\hat{\underset{\sim}{\Phi}}$, $\underset{\alpha}{\underset{\sim}{T}}$, $\underset{1}{\overset{+}{\underset{\sim}{m}}}$, $\underset{\alpha}{\underset{\sim}{h}}$, $\underset{1}{\overset{+}{v}}$, $\underset{\alpha}{g}$.

By results a. and b. in Section 6

$$\psi = \psi(\underset{2}{\rho_o}, \underset{\alpha}{\rho}, \operatorname{grad} \underset{\alpha}{\rho}, \vartheta, \underset{2}{k}, \underset{\alpha}{\nu}, \operatorname{grad} \underset{\alpha}{\nu}) \,. \tag{8.1}$$

However, by the assumption of frame-indifference, ψ depends on grad $\underset{\alpha}{\rho}$ and grad $\underset{\alpha}{\nu}$ through their inner products, which are nonlinear. Thus

$$\psi = \psi(\underset{2}{\rho_0}, \underset{1}{\rho}, \underset{2}{\rho}, \vartheta, \underset{2}{k}, \underset{1}{\nu}, \underset{2}{\nu}) . \tag{8.2}$$

By (6.10), linear representations for q, $\underset{1}{\overset{+}{\underset{\sim}{m}}}$, and $\hat{\underset{\sim}{\phi}}$ are

$$\underset{\sim}{q} = -\kappa_T \text{ grad } \vartheta - \Sigma \underset{\alpha}{\kappa}_{\rho} \text{ grad } \underset{\alpha}{\rho} - \kappa_D \underset{\sim}{v} - \Sigma \underset{\alpha}{\kappa}_{\nu} \text{ grad } \underset{\alpha}{\nu} , \tag{8.3}$$

$$\underset{1}{\overset{+}{\underset{\sim}{m}}} = -m_T \text{ grad } \vartheta - \Sigma \underset{\alpha}{m}_{\rho} \text{ grad } \underset{\alpha}{\rho} - m_D \underset{\sim}{v} - \Sigma \underset{\alpha}{m}_{\nu} \text{ grad } \underset{\alpha}{\nu} , \tag{8.4}$$

$$\hat{\underset{\sim}{\phi}} = -K_T \text{ grad } \vartheta - \Sigma \underset{\alpha}{K}_{\rho} \text{ grad } \underset{\alpha}{\rho} - K_D \underset{\sim}{v} - \Sigma \underset{\alpha}{K}_{\nu} \text{ grad } \underset{\alpha}{\nu} , \tag{8.4}$$

where the coefficients are functions of $\underset{2}{\rho_0}, \underset{\alpha}{\rho}, \vartheta, \underset{2}{k}, \underset{\alpha}{\nu}, \underset{\alpha}{\dot{\nu}}$. The representation (8.5) for $\hat{\underset{\sim}{\phi}}$ is further restricted by (6.47), (6.48), (6.53) and (8.2). Straightforward computation leads to the conclusion that K_T, $\underset{\alpha}{K}_{\rho}$, $\underset{\alpha}{K}_{\nu}$ vanish, giving

$$\hat{\underset{\sim}{\phi}} = -K_D \underset{\sim}{v} . \tag{8.6}$$

In this case (6.49) and (6.50) are satisfied identically. It is seen that, unlike the special case of a single granular material of Goodman and Cowin [2], the entropy flux does not have the classical form of the heat flux multiplied by the coldness. It is in addition affected by the diffusion velocity, with coefficient depending on density, volume distribution and its velocity, coldness, and equilibrated inertia.

There is, furthermore, a representation for equilibrated stress $\underset{a}{\tilde{h}}$ derived from (6.10). However, $\underset{a}{\tilde{h}}$ are obtained from $\hat{\tilde{\phi}}$ as given by (8.6) and ϕ as given by (8.2). An easy computation yields

$$\vartheta \underset{1}{\tilde{h}} = -\frac{\partial K_D}{\partial \underset{1}{\nu}} \tilde{v} , \tag{8.7}$$

$$\vartheta \underset{2}{\tilde{h}} = -\left(\frac{\partial K_D}{\partial \underset{2}{\nu}} + \frac{\partial \xi \rho_{12}}{\rho} \grave{\nu}\right) \underset{2}{\tilde{v}} . \tag{8.8}$$

Thus the equilibrated stress vanishes when the diffusion velocity vanishes.

By (6.11) and (6.12), representations for $\underset{a}{\tilde{T}}$, $\underset{1}{\overset{+}{v}}$ and $\underset{a}{g}$ are

$$\underset{1}{\overset{a}{\tilde{T}}} = -\mu \tilde{\Omega} , \tag{8.9}$$

$$\underset{a}{\overset{s}{\tilde{T}}} - \underset{a}{\overset{s}{\tilde{T}}}{}^{o} = \Sigma(\underset{ab}{\lambda} \underset{b}{\grave{\nu}} + \underset{ab}{\xi} \operatorname{tr} \underset{b}{\tilde{D}})\tilde{1} + 2\Sigma \underset{ab}{\eta} \underset{b}{\tilde{D}} , \tag{8.10}$$

$$\underset{1}{\overset{+}{v}} - \underset{1}{\overset{+}{v}}{}^{o} = \Sigma(\underset{a}{\nu} \underset{a}{\grave{\nu}} + \underset{a}{\mu} \operatorname{tr} \underset{a}{\tilde{D}}) , \tag{8.11}$$

$$\underset{a}{g} - \underset{a}{g}{}^{o} = -\Sigma(\underset{ab}{\zeta} \underset{b}{\grave{\nu}} + \underset{ab}{\delta} \operatorname{tr} \underset{b}{\tilde{D}}) , \tag{8.12}$$

where superscript ^{o} denotes values in equilibrium.

Inserting (8.3) - (8.12) into (6.58), and noting (7.2) - (7.13), I obtain the following restrictions on the coefficients in the linear theory:

$$K_D = -\vartheta\rho\rho_1 \frac{\partial\psi}{\partial\rho_1} - \vartheta p_1 \, ,$$

$$K_D = \vartheta\rho\rho_2 \frac{\partial\psi}{\partial\rho_2} + \vartheta\rho_2 \, ,$$

$$\kappa_T \geq 0,$$

$$\kappa_{1\rho} = 0, \quad \kappa_{2\rho} = 0,$$

$$\vartheta\rho m_{1\rho} = -\frac{\partial K_D}{\partial\rho_1} - \vartheta\rho_2 \frac{\partial\psi}{\partial\rho_1} \, ,$$

$$\vartheta\rho m_{2\rho} = -\frac{\partial K_D}{\partial\rho_2} + \vartheta\rho_1 \frac{\partial\psi}{\partial\rho_2} \, , \tag{8.13}$$

$$\xi_{11} \geq 0, \quad \xi_{22} \geq 0, \quad \xi_{11}\,\xi_{22} - \frac{1}{4}(\xi_{12} + \xi_{21})^2 \geq 0,$$

$$\eta_{11} \geq 0, \quad \eta_{22} \geq 0, \quad \eta_{11}\,\eta_{22} - \frac{1}{4}(\eta_{12} + \eta_{21})^2 \geq 0,$$

$$\mu \geq 0,$$

$$2\vartheta m_D \geq 0 \, ,$$

$$2\kappa_T(2\vartheta\rho m_D) \geq \left(\frac{\partial K_D}{\partial\vartheta} + \kappa_D + \vartheta\rho m_T + \frac{p_2}{\vartheta}\frac{\rho_1}{\rho}\right)\left(\frac{\partial K_D}{\partial\vartheta} + \kappa_D + \vartheta\rho m_T + \frac{p_2}{\vartheta}\frac{\rho_1}{\rho}\right) .$$

Finally, the determinant

$$
\begin{vmatrix}
2\xi_{11} & \xi_{12}+\xi_{21} & \lambda_{11}+\delta_{11}-\rho\mu_1 & \lambda_{12}+\delta_{21}+\rho\mu_1 \\
\xi_{12}+\xi_{21} & 2\xi_{22} & \lambda_{21}+\delta_{12}-\rho\mu_2 & \lambda_{22}+\delta_{22}+\rho\mu_2 \\
\lambda_{11}+\delta_{11}-\rho\mu_1 & \lambda_{21}+\delta_{12}-\rho\mu_2 & 2(\zeta_{11}-\rho\upsilon_1) & \zeta_{12}+\zeta_{21}+\rho(\upsilon_2-\upsilon_1) \\
\lambda_{12}+\delta+\rho\mu_1 & \lambda_{22}+\delta_{22}+\rho\mu_2 & \zeta_{12}+\zeta_{21}+\rho(\upsilon_2-\upsilon_1) & 2(\zeta_{22}+\rho\upsilon_2)
\end{vmatrix}
\tag{8.14}
$$

must be positive semi-definite. The restrictions (8.13) are classical restrictions for a mixture of two fluids as given by Müller. The inequalities (8.14) include $(8.13)_7$ but are otherwise new to this theory.

The form of the entropy flux $\underset{\sim}{\phi}$ in the linear theory can be found. By $(8.13)_{1,2}$

$$K_D = \vartheta\underset{\sim}{v}\left[\rho_{12}\rho\left(\frac{\partial\psi}{\partial\rho_1} - \frac{\partial\psi}{\partial\rho_2}\right) + \left(\frac{\rho_2}{\rho}p_1 - \frac{\rho_2}{\rho}p_2\right)\right] . \tag{8.15}$$

Substituting (6.23), (7.6) and (8.6) into (5.12), and comparing with (8.15) substituted into (8.6) gives

$$\underset{\sim}{\phi} = \vartheta\underset{\sim}{q} - \vartheta\underset{\sim}{v}\left[\rho_{12}\rho\left(\frac{\partial\psi}{\partial\rho_1} - \frac{\partial\psi}{\partial\rho_2}\right) - \frac{\partial K_D}{\partial\grave{\nu}_1}(\grave{\nu}_1 - \grave{\nu}_2)\right] . \tag{8.16}$$

This generalizes Müller's (7.21).

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